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On the usefulness of modulation spaces in deformation quantization

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Abstract

We discuss the relevance to deformation quantization of Feichtinger's modulation spaces, especially of the weighted Sjöstrand classes $M_s^{\infty,1}(\mathbb{R}^{2n})$. These function spaces are good classes of symbols of pseudodifferential operators (observables). They have a widespread use in time–frequency analysis and related topics, but are not very well known in physics. It turns out that they are particularly well adapted to the study of the Moyal star product and of the star exponential.

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1. Introduction

It has become rather obvious since the 1990s that the theory of modulation spaces, which plays a key role in time–frequency analysis (TFA) and Gabor analysis, often allows us to prove in a rather pedestrian way results that are usually studied with methods of 'hard' analysis. These spaces, whose definition goes back to the seminal work [10, 11] of Feichtinger over the period 1980–1983 (also see [37]), are however not generally well known by physicists, even those working in the phase-space formulation of quantum mechanics (QM). This is unfortunate, especially since 'interdisciplinarity' has become so fashionable in Science; it is a perfect example of two disciplines living in mirror universes, since, conversely, many techniques which have proved to be successful in QM (for instance, symplectic geometry) are more or less ignored in TFA (to be fair, Folland's book [12] comes as close as possible to such an interdisciplinary program, but this book was written in the 1980s, and there has been much progress both in TFA and quantum mechanics since then).

This paper is a first (and modest) attempt toward the construction of bridges between quantum mechanics in phase space, more precisely deformation quantization, and these

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new and insufficiently exploited functional-analytic techniques; this is made possible using the fact that ordinary (Weyl) pseudo-differential calculus and deformation quantization are 'intertwined' using the notion of wave-packet transform, as we have shown in our recent paper [20], and the fact that these wave-packet transforms are closely related to the windowed short-time Fourier transform appearing in the definition of modulation spaces.

This work is structured as follows:

- In section 2, we briefly review deformation quantization with an emphasis on the point of view developed in de Gosson and Luef [20]; in this approach the star product is expressed as the action of a pseudo-differential operator $\widetilde{A}^{\hbar}$ of a certain type ('Bopp operator'). In fact, the Moyal product $A \star_{\hbar} B$ of two observables can be expressed as
$$A \star_{\hbar} B = \widetilde{A}^{\hbar}(B). \tag{1}$$
That operator is related to the usual Weyl operator by an intertwining formula involving 'windowed wave-packet transforms', which are closely related to the short-time Fourier transform familiar from time–frequency analysis. We take the opportunity to comment a recent statement of Gerstenhaber on the choice of a 'preferred quantization'.
- In section 3, we begin by introducing the basics of the theory of modulation spaces we will need. We first define the modulation spaces $M_s^q(\mathbb{R}^n)$ which are particularly convenient for describing phase-space properties of wavefunctions. The use of modulation spaces in deformation quantization requires a redefinition of these spaces in terms of the cross-Wigner transform. We do not consider here the slightly more general spaces $M_s^{q,r}(\mathbb{R}^n)$, this mainly for the sake of notational brevity, however most of our results can be generalized without difficulty to this case. We thereafter introduce the weighted spaces $M_s^{\infty,1}(\mathbb{R}^{2n})$, which generalize the so-called Sjöstrand classes. The elements of these spaces are very convenient as pseudo-differential symbols (or 'observables'); we show that, in particular, $M_s^{\infty,1}(\mathbb{R}^{2n})$ is a $*$-algebra for the Moyal product (proposition 10): if $A, B \in M_s^{\infty,1}(\mathbb{R}^{2n})$, then $A \star_{\hbar} B \in M_s^{\infty,1}(\mathbb{R}^{2n})$ and $\overline{A} \in M_s^{\infty,1}(\mathbb{R}^{2n})$. We moreover prove the following 'Wiener property' of the Moyal product: if $A \in M_s^{\infty,1}(\mathbb{R}^{2n})$ and $A \star_{\hbar} B = I$ then $B \in M_s^{\infty,1}(\mathbb{R}^{2n})$. We finally redefine the star exponential
$$\exp(Ht) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{t}{\mathrm{i}\hbar}\right)^k \widetilde{H}^k, \tag{2}$$
in terms of the Bopp operators; in fact, we have
$$\exp(Ht) = \exp\left(-\frac{\mathrm{i}}{\hbar}\widetilde{H}t\right). \tag{3}$$
This allows us to prove regularity results for $\exp(Ht)$.

1.1. Notation

The scalar product of two square integrable functions ψ and ψ' on $\mathbb{R}^n$ is written as $(\psi|\psi')$; that of functions Ψ, Ψ' on $\mathbb{R}^{2n}$ is $((\Psi|\Psi'))$. We denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of functions decreasing, together with their derivatives, faster than the inverse of any polynomial. The dual $\mathcal{S}'(\mathbb{R}^n)$ of $\mathcal{S}(\mathbb{R}^n)$ is the space of tempered distributions. The standard symplectic form on $\mathbb{R}^n \times \mathbb{R}^n \equiv \mathbb{R}^{2n}$ is given by $\sigma(z, z') = p \cdot x' - p' \cdot x$ if $z = (x, p)$ and $z' = (x', p')$; equivalently, $\sigma(z, z') = Jz \cdot z'$ where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ is the standard symplectic matrix. When using matrix notation x, p, z are viewed as column vectors.

If A is a 'symbol' we denote indifferently by $A^w(x, -\mathrm{i}\hbar\partial_x)$ or $\widehat{A}^{\hbar}$ the corresponding Weyl operator.

We will also use multi-index notation: for $\alpha = (\alpha_1, \ldots, \alpha_{2n})$ in $\mathbb{N}^{2n}$ we set

$$|\alpha| = \alpha_1 + \cdots + \alpha_{2n}, \qquad \partial_z^\alpha = \partial_{z_1}^{\alpha_1} \cdots \partial_{z_{2n}}^{\alpha_{2n}}$$

where $\partial_{z_j}^{\alpha_j} = \partial^{\alpha_j} / \partial x_j^{\alpha_j}$ for $1 \leqslant j \leqslant n$ and $\partial_{z_j}^{\alpha_j} = \partial^{\alpha_j} / \partial \xi_j^{\alpha_j}$ for $n+1 \leqslant j \leqslant 2n$.

The unitary $\hbar$-Fourier transform is defined, for $\psi \in \mathcal{S}(\mathbb{R}^n)$, by

$$F\psi(x) = \left(\frac{1}{2\pi\hbar}\right)^{n/2} \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} x \cdot x'} \psi(x')\, \mathrm{d}x'.$$

2. Deformation quantization and Bopp calculus

The rigorous definition of deformation quantization goes back to the work [1, 2] of Bayen *et al* in the end of the 1970s; also see the contribution by Maillard [30]. We recommend the reading of Sternheimer's paper [36] for a recent discussion of the topic and its genesis. The relation with a special Weyl calculus (which we call 'Bopp calculus') was introduced in de Gosson and Luef.

2.1. Deformation quantization

2.1.1. Generalities. Roughly speaking, the starting idea is that if we view classical mechanics as the limit of quantum mechanics when $\hbar \to 0$, then we should be able to construct quantum mechanics by 'deforming' classical mechanics. On the simplest level (which is the one considered in this paper), one replaces the ordinary product of two functions on phase space, say A and B, by a 'star product'

$$A \star_\hbar B = AB + \sum_{j=1}^{\infty} \hbar^j C_j(A, B)$$

where C_j are certain bidifferential operators. Since one wants the star product to define an algebra structure, one imposes certain conditions on $A \star_\hbar B$: it should be associative; moreover, it should become the ordinary product AB in the limit $\hbar \to 0$ and we should recover the Poisson bracket $\{A, B\}$ from the quantity $\mathrm{i}\hbar^{-1}(A \star_\hbar B - B \star_\hbar A)$ when $\hbar \to 0$.

Assume now that

$$\widehat{A}^\hbar = A^w(x, -\mathrm{i}\hbar\partial_x) : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$$

and

$$\widehat{B}^\hbar = B^w(x, -\mathrm{i}\hbar\partial_x) : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n).$$

Then the product $\widehat{C}^\hbar = \widehat{A}^\hbar \widehat{B}^\hbar$ is defined on $\mathcal{S}(\mathbb{R}^n)$ and we have $\widehat{C}^\hbar = C^w(x, -\mathrm{i}\hbar\partial_x)$ where the symbol C is given by the Moyal product $C = A \star_\hbar B$:

$$A \star_\hbar B(z) = \left(\frac{1}{4\pi\hbar}\right)^{2n} \int\int_{\mathbb{R}^{2n}} \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}\sigma(u,v)} A\left(z + \frac{1}{2}u\right) B\left(z - \frac{1}{2}v\right) \mathrm{d}u\, \mathrm{d}v \qquad (4)$$

(historically formula (4) goes back to the seminal work of Moyal [31] and Groenewold [26]). Equivalently

$$A \star_\hbar B(z) = \left(\frac{1}{\pi\hbar}\right)^{2n} \int\int_{\mathbb{R}^{2n}} \mathrm{e}^{-\frac{2\mathrm{i}}{\hbar}\sigma(z-z', z-z'')} A(z') B(z'')\, \mathrm{d}z'\, \mathrm{d}z''. \qquad (5)$$

Recall that the Weyl symbol of an operator $\widehat{A}^\hbar : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ is the distribution $A \in \mathcal{S}'(\mathbb{R}^{2n})$ such that

$$\widehat{A}^\hbar = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} A_\sigma(z_0) \widehat{T}^\hbar(z_0)\, \mathrm{d}z_0, \qquad (6)$$

where $\widehat{T}^{\hbar}(z_0)$ is the Heisenberg–Weyl operator, defined by

$$\widehat{T}^{\hbar}(z_0)\psi(x) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(p_0\cdot x - \frac{1}{2}p_0\cdot x_0)}\psi(x-x_0) \tag{7}$$

if $z_0 = (x_0, p_0)$ and

$$A_\sigma(z) = F_\sigma A(z) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\sigma(z,z')}A(z')\,\mathrm{d}z \tag{8}$$

is the symplectic Fourier transform of A; note that $F_\sigma A(z) = FA(-Jz)$.

It is clear that the Moyal product is associative (because composition of operators is); to see that $\lim_{\hbar\to 0} A \star_\hbar B = AB$ it suffices (at least on a formal level) to perform the change of variables $(u, v) \longmapsto \sqrt{\hbar}(u, v)$ in the integral in (4), which leads to

$$A \star_\hbar B(z) = \left(\frac{1}{4\pi}\right)^{2n} \int\int_{\mathbb{R}^{2n}} \mathrm{e}^{\frac{\mathrm{i}}{2}\sigma(u,v)} A\left(z + \frac{\sqrt{\hbar}}{2}u\right) B\left(z - \frac{\sqrt{\hbar}}{2}v\right) \mathrm{d}u\,\mathrm{d}v, \tag{9}$$

letting $\hbar \to 0$ and using the Fourier inversion formula

$$\int\int_{\mathbb{R}^{2n}} \mathrm{e}^{\frac{\mathrm{i}}{2}\sigma(u,v)}\,\mathrm{d}u\,\mathrm{d}v = (4\pi)^{2n}.$$

we get $\lim_{\hbar\to 0} C(z) = A(z)B(z)$. That we also have

$$\lim_{\hbar\to 0}[\mathrm{i}\hbar^{-1}(A \star_\hbar B - B \star_\hbar A)] = \{A, B\}$$

is verified in a similar way.

2.1.2. The notion of the Bopp pseudo-differential operator. There is another way to write the Moyal product, which is reminiscent of formula (6) for Weyl pseudodifferential operators. Performing the change of variables $v = z_0$, $z + \frac{1}{2}u = z'$ in formula (4) we get

$$\begin{aligned} A \star_\hbar B(z) &= \left(\frac{1}{2\pi\hbar}\right)^{2n} \int\int_{\mathbb{R}^{4n}} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\sigma(z_0,z'-z)}A(z')B\left(z - \frac{1}{2}z_0\right)\mathrm{d}z_0\,\mathrm{d}z' \\ &= \left(\frac{1}{2\pi\hbar}\right)^{2n} \int_{\mathbb{R}^{2n}} \left[\int_{\mathbb{R}^{2n}} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\sigma(z_0,z')}A(z')\,\mathrm{d}z'\right] \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\sigma(z_0,z)} B\left(z - \frac{1}{2}z_0\right)\mathrm{d}z_0. \end{aligned}$$

Defining the operators $\widetilde{T}(z_0) : \mathcal{S}(\mathbb{R}^{2n}) \longrightarrow \mathcal{S}(\mathbb{R}^{2n})$ by

$$\widetilde{T}(z_0)B(z) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\sigma(z_0,z)}B\left(z - \tfrac{1}{2}z_0\right) \tag{10}$$

we can thus write the Moyal product in the form

$$A \star_\hbar B = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} A_\sigma(z_0)(\widetilde{T}(z_0)B)\,\mathrm{d}z_0. \tag{11}$$

This formula, which is reminiscent of the representation (6) of Weyl operators, will play an important role in the subsequent sections. Note that the operators $\widetilde{T}(z_0)$ are unitary on $L^2(\mathbb{R}^{2n})$ and satisfy the same commutation relations as the Heisenberg–Weyl operators.

In [20] we have proven the following results:

Proposition 1. *The Weyl symbol of the operator*

$$\widetilde{A}^{\hbar} : B \longmapsto \widetilde{A}^{\hbar}(B) = A \star_\hbar B \tag{12}$$

is the distribution $\mathbb{A} \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ *given by*

$$\mathbb{A}(z, \zeta) = A\left(z - \tfrac{1}{2}J\zeta\right) = A\left(x - \tfrac{1}{2}\zeta_p,\, p + \tfrac{1}{2}\zeta_x\right), \tag{13}$$

where $z \in \mathbb{R}^{2n}$ *and* $\zeta \in \mathbb{R}^{2n}$ *are viewed as dual variables.*

2.1.3. Symplectic covariance. Recall that the metaplectic group $\mathrm{Mp}(2n, \mathbb{R})$ is the unitary representation of the connected double covering of the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ (see e.g. [12, 16, 29]). The metaplectic group is generated by the following unitary operators:

- the modified $\hbar$-Fourier transform
$$\widehat{J}^{\hbar} = \mathrm{i}^{-n/2} F \tag{14}$$
whose projection on $\mathrm{Sp}(2n, \mathbb{R})$ is the standard symplectic matrix J;
- the 'chirps' $\widehat{V_{-P}}^{\hbar}$ defined, for $P = P^T$ by
$$\widehat{V_{-P}}^{\hbar} \psi(x) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar} Px\cdot x} \psi(x) \tag{15}$$
whose projection on $\mathrm{Sp}(2n, \mathbb{R})$ is $\begin{pmatrix} I & 0 \\ P & I \end{pmatrix}$;
- the unitary changes of variables, defined for invertible L by
$$\widehat{M_{L,m}}^{\hbar} \psi(x) = \mathrm{i}^m \sqrt{|\det L|} \psi(Lx), \tag{16}$$
where the integer m corresponds to a choice of $\arg\det L$; its projection on $\mathrm{Sp}(2n, \mathbb{R})$ is $\begin{pmatrix} L^{-1} & 0 \\ 0 & L^T \end{pmatrix}$.

Every $S \in \mathrm{Sp}(2n, \mathbb{R})$ is the projection of two operators $\pm\widehat{S}^{\hbar}$ in $\mathrm{Mp}(2n, \mathbb{R})$.
We recall the following fundamental symplectic covariance property of Weyl calculus:
$$\widehat{(A \circ S^{-1})}^{\hbar} = \widehat{S}^{\hbar} \widehat{A}^{\hbar} \widehat{S}^{\hbar -1}, \tag{17}$$
where $\widehat{S}^{\hbar}$ is any of the two metaplectic operators associated with S.

Proposition 2. *For every $S \in \mathrm{Sp}(2n, \mathbb{R})$ we have*
$$(A \circ S^{-1})\star_{\hbar} = U_S (A\star_{\hbar}) U_S^{-1}, \tag{18}$$
where U_S is the unitary operator on $L^2(\mathbb{R}^{2n})$ defined by $U_S\Psi(z) = \Psi(Sz)$, and we have $U_S \in \mathrm{Mp}(4n, \mathbb{R})$.

Proof. To prove (18) we note that $A\star_{\hbar}$ is the Bopp operator with Weyl symbol $\mathbb{A}(z, \zeta) = A\left(z - \frac{1}{2}J\zeta\right)$. Let $\widetilde{A}^{S^{-1}}$ be the Weyl symbol of the Bopp operator $\widetilde{H \circ S^{-1}}$; since $S^{-1}J = JSS^T$ we have
$$(\widetilde{A}^{\hbar})^{S^{-1}}(z, \zeta) = A\left(S^{-1}\left(z - \tfrac{1}{2}J\zeta\right)\right) = \widetilde{A}^{\hbar}(M_S(z, \zeta))$$
with
$$M_S = \begin{pmatrix} S^{-1} & 0 \\ 0 & S^T \end{pmatrix} \in \mathrm{Sp}(4n, \mathbb{R}). \tag{19}$$
($\mathrm{Sp}(4n, \mathbb{R})$ is the symplectic group of $\mathbb{R}^{4n}$ equipped with the standard symplectic form $\sigma \oplus \sigma$). It follows from the general theory of the metaplectic group (see in particular proposition 7.8(i) in [16]) that M_S is the projection on $\mathrm{Sp}(4n, \mathbb{R})$ of the metaplectic operator U_S defined by
$$U_S\Psi(z) = \sqrt{\det S}\,\Psi(Sz) = \Psi(Sz)$$
(recall that $\det S = 1$). This proves (18) applying the covariance formula (17) to $\widetilde{H}$ viewed as a Weyl operator. That $U_S \in \mathrm{Mp}(4n, \mathbb{R})$ is clear (cf formula (16)). □

2.1.4. On the use of Weyl calculus in deformation quantization. We take the opportunity to briefly discuss a remark done by Gerstenhaber in his recent paper [13]. The Weyl correspondence resolves in a particular way the ordering ambiguity when one passes from a symbol ('classical observable') $A(x, p)$ to its quantized version $A(\widehat{x}, \widehat{p})$; for instance, to monomials such as xp or x^2p it associates the symmetrized operators $\frac{1}{2}(\widehat{x}\widehat{p} + \widehat{p}\widehat{x})$ and $\frac{1}{3}(\widehat{x}^2\widehat{p} + \widehat{x}\widehat{p} + \widehat{p}\widehat{x}^2)$. This choice, argues Gerstenhaber, is totally arbitrary and other choices are, *a priori*, equally good (for instance, people working in partial differential equations would usually choose the quantizations $\widehat{x}\widehat{p}$ and $\widehat{x}^2\widehat{p}$ in the examples above), in fact for a given symbol we have infinitely many choices

$$\widehat{A}_\tau^\hbar \psi(x) = \left(\frac{1}{2\pi\hbar}\right)^n \int\int_{\mathbb{R}^{2n}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}p\cdot(x-y)} A((1-\tau)x + \tau y, p)\psi(y)\,\mathrm{d}y\,\mathrm{d}p \tag{20}$$

corresponding to a parameter value $\tau \in [0, 1]$ (see [33]); Weyl quantization corresponds to the choice $\tau = 1/2$. Gerstenhaber is right, no doubt. However, one should understand that when working in deformation quantization, the Weyl correspondence is still the most 'natural', and this for the following reason: the primary aim of deformation quantization is to view quantum mechanics as a deformation of a classical theory, namely classical mechanics in its Hamiltonian formulation. Now, one of the main features of the Hamiltonian approach is its symplectic covariance. It is therefore certainly desirable that the objects that one introduces in a theory whose vocation is to mimic Hamiltonian mechanics retains this fundamental feature. It turns out that not only is Weyl calculus a symplectically covariant theory, but it is also the *only* quantization scheme having this property! This fact, which was already known to Shale [32] (and is proven in detail in the last chapter of Wong's book [38]) justifies *a posteriori* the suitability of the Weyl correspondence in deformation quantization, as opposed to other ordering schemes.

2.2. The Moyal product and Bopp operators

2.2.1. Windowed wave-packet transforms. For $\phi \in L^2(\mathbb{R}^n)$ such that $\|\phi\|_{L^2} = 1$ we define the windowed wave-packet transform $W_\phi : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^{2n})$ by

$$W_\phi\psi = (2\pi\hbar)^{n/2} W(\psi, \phi) \tag{21}$$

for $\psi \in \mathcal{S}'(\mathbb{R}^n)$; here $W(\psi, \phi)$ is the usual cross-Wigner transform, given by

$$W(\psi, \phi)(z) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}p\cdot y}\psi\left(x + \frac{1}{2}y\right)\overline{\phi\left(x - \frac{1}{2}y\right)}\,\mathrm{d}y. \tag{22}$$

The windowed wave-packet transform is thus explicitly given by

$$W_\phi\psi(z) = \left(\frac{1}{2\pi\hbar}\right)^{n/2} \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}p\cdot y}\psi\left(x + \frac{1}{2}y\right)\overline{\phi\left(x - \frac{1}{2}y\right)}\,\mathrm{d}y.$$

Since $\|\phi\|_{L^2} = 1$ it follows from Moyal's identity

$$((W(\psi, \phi)|W(\psi', \phi'))) = \left(\frac{1}{2\pi\hbar}\right)^n (\psi|\psi')\overline{(\phi|\phi')} \tag{23}$$

(see e.g. [16, 22]) that the restriction of W_ϕ to $L^2(\mathbb{R}^n)$ is a linear isometry of $L^2(\mathbb{R}^n)$ onto a subspace $\mathcal{H}_\phi$ of $L^2(\mathbb{R}^{2n})$. A simple calculation shows that for $\Psi \in \mathcal{S}(\mathbb{R}^n)$ the adjoint $W_\phi^* : L^2(\mathbb{R}^{2n}) \longrightarrow L^2(\mathbb{R}^n)$ of W_ϕ is given by

$$W_\phi^*\Psi(x) = \left(\frac{2}{\pi\hbar}\right)^{n/2} \int_{\mathbb{R}^n} \mathrm{e}^{\frac{2\mathrm{i}}{\hbar}p\cdot(x-y)}\phi(2y - x)\Psi(y, p)\,\mathrm{d}p\,\mathrm{d}y. \tag{24}$$

The subspace $\mathcal{H}_\phi$ is closed (and hence a Hilbert space): the mapping $P_\phi = W_\phi W_\phi^*$ satisfies $P_\phi = P_\phi^*$ and $P_\phi P_\phi^* = P_\phi$ hence P_ϕ is an orthogonal projection. Since $W_\phi^* W_\phi$ is the identity on $L^2(\mathbb{R}^n)$ the range of W_ϕ^* is $L^2(\mathbb{R}^n)$ and that of P_ϕ is therefore precisely $\mathcal{H}_\phi$. Since $\mathcal{H}_\phi$ is the range of P_ϕ and the closedness of $\mathcal{H}_\phi$ follows.

We close with a remark that the window-wave-packet transform is related to the short-time Fourier transform (STFT) with window $\phi \in \mathcal{S}(\mathbb{R}^n)$:

$$V_\phi \psi(z) = \int_{\mathbb{R}^n} \mathrm{e}^{-2\pi \mathrm{i} p\cdot x'} \psi(x') \overline{\phi(x'-x)}\,\mathrm{d}x'. \tag{25}$$

The STFT is a standard tool in time–frequency analysis to study the phase-space content of functions and distributions [22]. The precise relation between the STFT and the windowed-wave-packet transform is given by the formula:

$$W_\phi \psi(z) = 2^n\, \mathrm{e}^{\frac{2\mathrm{i}}{\hbar} p\cdot x} V_{\phi^\vee_{\sqrt{2\pi\hbar}}} \psi_{\sqrt{2\pi\hbar}} \left(\sqrt{\frac{2}{\pi\hbar}} z \right), \tag{26}$$

where $\phi^\vee(x) = \phi(-x)$ and $\psi_{\sqrt{2\pi\hbar}}(x) = \psi(x\sqrt{2\pi\hbar})$. The connection between these two objects strongly suggests that results from time–frequency analysis should have some interesting consequences for problems in the phase-space formulation of quantum mechanics.

2.2.2. The intertwining property. The key to the relation between deformation quantization and Bopp calculus comes from following result:

Proposition 3. *We have the intertwining formulae*

$$\widetilde{A}^\hbar W_\phi = W_\phi \widehat{A}^\hbar, \qquad W_\phi^* \widetilde{A}^\hbar = \widehat{A}^\hbar W_\phi^* \tag{27}$$

where $W_\phi^* : \mathcal{S}(\mathbb{R}^{2n}) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ *is the adjoint of* W_ϕ. *Equivalently:*

$$A \star_\hbar (W_\phi \psi) = W_\phi(\widehat{A}^\hbar \psi), \qquad W_\phi^*(A \star_\hbar B) = \widehat{A}^\hbar (W_\phi^* B) \tag{28}$$

for $\psi \in \mathcal{S}(\mathbb{R}^n)$.

Proof. See proposition 2 in [20]. □

Formula (13) justifies the notation

$$\widetilde{A}^\hbar = A\left(x + \tfrac{1}{2}\mathrm{i}\hbar\partial_p, p - \tfrac{1}{2}\mathrm{i}\hbar\partial_x\right) \tag{29}$$

and we will call $\widetilde{A}^\hbar$ the *Bopp pseudo-differential operator with symbol* A; the terminology is inspired by the paper [4] by Bopp, who was apparently the first to suggest the use of the non-standard quantization rules

$$(x, p) \longmapsto \left(x + \tfrac{1}{2}\mathrm{i}\hbar\partial_p, p - \tfrac{1}{2}\mathrm{i}\hbar\partial_x\right) \tag{30}$$

(which also appear in Kubo's paper [28]). We note that formula (29) is found in many physical texts *without justification*. It was precisely one of the aims of [20] to give a rigorous justification of this notation.

3. Modulation spaces

We define and list the main properties of two particular types of modulation spaces: (i) the spaces $M_s^q(\mathbb{R}^n)$ which contain, as a particular case, the Feichtinger algebra $S_0(\mathbb{R}^d)$ and (ii) the spaces $M_s^{\infty,1}(\mathbb{R}^{2n})$ which are a generalization of the Sjöstrand classes. We refer to Gröchenig's book [22] for proofs and generalizations.

3.1. The modulation spaces $M_s^q(\mathbb{R}^n)$

The main idea underlying the construction of modulation spaces is to consider classes of functions or distributions where the phase-space representations fulfil some decay or integrability condition with respect to a window function. In our presentation, we use the windowed-wave-packet transform in the definition of the modulation spaces $M_s^q(\mathbb{R}^n)$ instead of the traditional used STFT. Recall that we observed that these two objections provide the same kind of phase-space content of a given function 26. For a finer analysis of the phase-space content of a function it is very useful to impose some weighted L^q-spaces conditions.

For our treatment of the Moyal product it turns out that the radial symmetric weight v_s on $\mathbb{R}^{2n}$

$$v_s(z) = (1 + |z|^2)^{s/2} \tag{31}$$

provides the correct class of modulation spaces. Note that v_s is submultiplicative

$$v_s(z + z') \leqslant v_s(z)v_s(z'). \tag{32}$$

In what follows q is a real number in $[0, \infty]$. Let $L_s^q(\mathbb{R}^{2n})$ be the space of all Lebesgue-measurable functions Ψ on $\mathbb{R}^{2n}$ such that $v_s\Psi \in L^q(\mathbb{R}^{2n})$. When $q < \infty$ the formula

$$\|\Psi\|_{L_s^q} = \left(\int_{\mathbb{R}^{2n}} |v_s(z)\Psi(z)|^q \,\mathrm{d}z\right)^{1/q}$$

defines a norm on $L_s^q(\mathbb{R}^{2n})$; in the case $q = \infty$ this formula is replaced by

$$\|\Psi\|_{L_s^\infty} = \operatorname*{ess\,sup}_{z\in\mathbb{R}^{2n}} |v_s(z)\Psi(z)|.$$

The modulation space $M_s^q(\mathbb{R}^n)$ is the vector space consisting of all $\psi \in \mathcal{S}'(\mathbb{R}^n)$ such that $W_\phi\psi \in L_s^q(\mathbb{R}^{2n})$ (or $V_\phi\psi \in L_s^q(\mathbb{R}^{2n})$. We thus have $\psi \in M_s^q(\mathbb{R}^n)$ if and only if there exists $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\|\psi\|_{M_s^q}^\phi = \left(\int_{\mathbb{R}^{2n}} |v_s(z)W_\phi\psi(z)|^q \,\mathrm{d}z\right)^{1/q} < \infty \tag{33}$$

when $q < \infty$, and

$$\|\psi\|_{M_s^\infty}^\phi = \operatorname*{ess\,sup}_{z\in\mathbb{R}^{2n}} |v_s(z)W_\phi\psi(z)| < \infty \tag{34}$$

when $q = \infty$. Note that this definition is independent of the choice of the 'window' ϕ, and $\|\cdot\|_{M_s^q}^\phi$ for $\phi \in \mathcal{S}(\mathbb{R}^n)$ form a family of equivalent norms for the Banach space $M_s^q(\mathbb{R}^n)$. Moreover, the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is dense in $M_s^q(\mathbb{R}^n)$.

The class of modulation spaces $M_s^q(\mathbb{R}^n)$ contain as particular cases many of the classical function spaces. For instance, $M_s^2(\mathbb{R}^n)$ coincides with the Shubin–Sobolev space

$$Q^s(\mathbb{R}^{2n}) = L_s^2(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$$

([33], p 45). We also have that the Schwartz space has a description in terms of the windowed-wave-packet transform, e.g.

$$\mathcal{S}(\mathbb{R}^n) = \bigcap_{s\geqslant 0} M_s^2(\mathbb{R}^n).$$

The modulation spaces $M_s^q(\mathbb{R}^n)$ have two remarkable invariance properties.

Proposition 4. *(i) Each space $M_s^q(\mathbb{R}^n)$ is invariant under the action of the Heisenberg–Weyl operators $\widehat{T}^\hbar(z)$; in fact, there exists a constant $C > 0$ such that*

$$\|\widehat{T}^\hbar(z)\psi\|_{M_s^q}^\phi \leqslant C v_s(z)\|\psi\|_{M_s^q}^\phi. \tag{35}$$

(ii) For $1 \leqslant q < \infty$ the space $M_s^q(\mathbb{R}^n)$ is invariant under the action of the metaplectic group $\mathrm{Mp}(2n, \mathbb{R})$*: if $\widehat{S}^\hbar \in \mathrm{Mp}(2n, \mathbb{R})$ then $\widehat{S}^\hbar\psi \in M_s^q(\mathbb{R}^n)$ if and only if $\psi \in M_s^q(\mathbb{R}^n)$. In particular, $M_s^q(\mathbb{R}^n)$ is invariant under the Fourier transform.*

The proof of these facts is based on properties of the windowed-wave-packet transform, which allows one to translate the statement into problems for $L_s^q(\mathbb{R}^n)$. By the following property of $W_\phi\psi$:

$$\begin{aligned} W(\widehat{T}^\hbar(z_0)\psi, \phi)(z) &= T(z_0)W(\psi, \phi)(z) \\ &= W(\psi, \phi)(z - z_0) \end{aligned}$$

(i) is equivalent to the translation invariance of $L_s^q(\mathbb{R}^n)$, which is an elementary consequence of the submultiplicativity of the weight v_s. The covariance properties of the cross-Wigner transform with respect to metaplectic representation yield in a similar way the invariance of $M_s^q(\mathbb{R}^n)$ under metaplectic transforms. For later reference we want to point out that this implies in particular its invariance under rescalings.

Corollary 5. *The modulation space $M_s^q(\mathbb{R}^n)$ is invariant under the rescalings $\psi \longmapsto \psi_\lambda$ where $\psi_\lambda(x) = \psi(\lambda x)$ and where $\lambda \neq 0$. More generally, $M_s^q(\mathbb{R}^n)$ is invariant under every change of variables $x \longmapsto Lx$* $(\det L \neq 0)$.

A particularly interesting example of modulation space is obtained by taking $q = 1$ and $s = 0$; the corresponding space $M_0^1(\mathbb{R}^n)$ is often denoted by $S_0(\mathbb{R}^n)$, and is called the *Feichtinger algebra* [10]. $S_0(\mathbb{R}^n)$ is an algebra both for pointwise product and for convolution. We have the inclusions

$$\mathcal{S}(\mathbb{R}^n) \subset S_0(\mathbb{R}^n) \subset C^0(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n). \tag{36}$$

A remarkable property of the Feichtinger algebra is that is the smallest Banach space invariant under the action of the Heisenberg–Weyl operators (7):

Proposition 6. *Let $(\mathcal{B}, \|\cdot\|)$ be a Banach algebra of tempered distributions on $\mathbb{R}^n$. Suppose that $\mathcal{B}$ satisfies the two following conditions: (i) there exists $C > 0$ such that*

$$\|\widehat{T}^\hbar(z)\psi\| \leqslant C v_s(z)\|\psi\|$$

for all $z \in \mathbb{R}^{2n}$ and $\psi \in \mathcal{B}$; (ii) $M_s^1(\mathbb{R}^n) \cap \mathcal{B} \neq \{0\}$. Then $M_s^1(\mathbb{R}^n)$ is embedded in $\mathcal{B}$ and $S_0(\mathbb{R}^n) = M_0^1(\mathbb{R}^n)$ is the smallest algebra having this property.

(See [22], theorem 12.1.9, for a proof.)

The Feichtinger algebra $S_0(\mathbb{R}^n)$ contains non-differentiable functions, such as

$$\psi(x) = \begin{cases} 1 - |x|, & \text{if} \quad |x| \leqslant 1 \\ 0, & \text{if} \quad |x| > 1 \end{cases}$$

and it is thus a more general space than the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. This property, together with the fact that Banach spaces are mathematically easier to deal with than Fréchet spaces, makes the Feichtinger algebra into a tool of choice for the study of wavepackets.

3.2. A good symbol (=observable) class: $M_s^{\infty,1}(\mathbb{R}^{2n})$

In the 1970s the study of L^2-boundedness of pseudo-differential operators was a popular area of research. For instance, a landmark was the proof by Calderón and Vaillancourt [5] that every operator with symbol in $C^{2n+1}(\mathbb{R}^{2n})$ satisfying an additional condition had this property (the same applies to the Hörmander class $S_{0,0}^0(\mathbb{R}^{2n})$). It turns out that results of this type—whose proofs needed methods from hard analysis—are much better understood (and easier proved) using the theory of modulation spaces. For instance, Calderón and Vaillancourt's theorem is a simple corollary of the theory of the modulation space.

In [34, 35], Sjöstrand introduced a class of symbols which was shown by Gröchenig [24] to be identical with the modulation space

$$M_0^{\infty,1}(\mathbb{R}^{2n}) = M^{\infty,1}(\mathbb{R}^{2n}).$$

The Sjöstrand class $M^{\infty,1}(\mathbb{R}^{2n})$ contains, in particular, the symbol class $S_{0,0}^0(\mathbb{R}^{2n})$ consisting of all infinitely differentiable complex functions A on $\mathbb{R}^{2n}$ such that $\partial_z^\alpha A$ is bounded for all multi-indices $\alpha \in \mathbb{N}^{2n}$.

In this section, we study a weighted version of the Sjöstrand class for the weight function v_s on $\mathbb{R}^{4n}$

$$v_s(z,\zeta) = (1+|z|^2+|\zeta|^2)^{s/2} \tag{37}$$

(some of the results we list below remain valid for more general weight functions). By definition, $M_s^{\infty,1}(\mathbb{R}^{2n})$ consists of all $A \in \mathcal{S}'(\mathbb{R}^{2n})$ such that there exists a function $\Phi \in \mathcal{S}(\mathbb{R}^{2n})$ for which

$$\sup_{z\in\mathbb{R}^{2n}} [|V_\Phi A(z,\cdot)| v_s(z,\cdot)] \in L_s^1(\mathbb{R}^{2n}), \tag{38}$$

where $V_\Phi A$ is the short-time Fourier transform of A windowed by $\Phi \in \mathcal{S}(\mathbb{R}^{2n})$:

$$V_\Phi A(z,\zeta) = \int_{\mathbb{R}^{2n}} \mathrm{e}^{-2\pi \mathrm{i}\zeta\cdot z'} A(z')\overline{\Phi(z'-z)}\,\mathrm{d}z'. \tag{39}$$

The formula

$$\|A\|_{M_s^{\infty,1}}^{\Phi} = \int_{\mathbb{R}^{2n}} \sup_{z\in\mathbb{R}^{2n}} [|V_\Phi A(z,\zeta)| v_s(z,\zeta)]\,\mathrm{d}\zeta < \infty \tag{40}$$

defines a norm on $M_s^{\infty,1}(\mathbb{R}^{2n})$. A remarkable (and certainly not immediately obvious!) fact is that if condition (40) holds for one window Φ, then it holds for all windows; moreover, when Φ runs through $\mathcal{S}(\mathbb{R}^{2n})$ the functions $\|\cdot\|_{M_s^{\infty,1}}^{\Phi}$ form a family of equivalent norms on $M_s^{\infty,1}(\mathbb{R}^{2n})$. It turns out that $M_s^{\infty,1}(\mathbb{R}^{2n})$ is a Banach space for the topology defined by any of these norms; moreover, the Schwartz space $\mathcal{S}(\mathbb{R}^{2n})$ is dense in $M_s^{\infty,1}(\mathbb{R}^{2n})$.

In particular, the Sjöstrand class $M^{\infty,1}(\mathbb{R}^{2n})$ thus consists of all $A \in \mathcal{S}'(\mathbb{R}^{2n})$ such that

$$\int_{\mathbb{R}^{2n}} \sup_{z\in\mathbb{R}^{2n}} |V_\Phi A(z,\zeta)|\,\mathrm{d}\zeta < \infty$$

for some window $\Phi \in \mathcal{S}(\mathbb{R}^{2n})$.

The spaces $M_s^{\infty,1}(\mathbb{R}^{2n})$ are invariant under linear changes of variables:

Proposition 7. *Let M be a real invertible $2n \times 2n$ matrix. If $A \in M_s^{\infty,1}(\mathbb{R}^{2n})$ then $A \circ M \in M_s^{\infty,1}(\mathbb{R}^{2n})$. In fact, there exists a constant $C_M > 0$ such that for every window Φ and every $A \in M_s^{\infty,1}(\mathbb{R}^{2n})$ we have*

$$\|A\circ M\|_{M_s^{\infty,1}}^{\Phi} \leqslant C_M \|A\|_{M_s^{\infty,1}}^{\Psi} \tag{41}$$

where $\Psi = \Phi \circ M^{-1}$.

Proof. It suffices to prove the estimate (41) since $A \in M_s^{\infty,1}(\mathbb{R}^{2n})$ if and only if $\|A\|_{M_s^{\infty,1}}^{\Psi} < \infty$. Let us set $B = A \circ M$; performing the change of variables $z' \longmapsto Mz'$ we have

$$V_\Phi B(z,\zeta) = (\det M)^{-1} \int_{\mathbb{R}^{2n}} \mathrm{e}^{-2\pi \mathrm{i}\zeta\cdot M^{-1}z'} A(z')\overline{\Phi(M^{-1}z'-z)}\,\mathrm{d}z'$$

and hence

$$V_\Phi B(M^{-1}z, M^T\zeta) = (\det M)^{-1}\int_{\mathbb{R}^{2n}} \mathrm{e}^{-2\pi\mathrm{i}\zeta\cdot z'} A(z')\overline{\Phi(M^{-1}(z'-z))}\,\mathrm{d}z'$$

that is

$$V_\Phi B(z,\zeta) = (\det M)^{-1} V_\Psi A(Mz, (M^T)^{-1}\zeta), \qquad \Psi = \Phi\circ M^{-1}.$$

It follows that

$$\sup_{z\in\mathbb{R}^{2n}}[|V_\Phi B(z,\zeta)|v_s(z,\zeta)] = (\det M)^{-1}\sup_{z\in\mathbb{R}^{2n}}[V_\Psi A(z,(M^T)^{-1}\zeta)v_s(M^{-1}z,\zeta)]$$

so that

$$\begin{aligned}\|B\|^{\Phi}_{M_s^{\infty,1}} &= (\det M)^{-1}\int_{\mathbb{R}^{2n}}\sup_{z\in\mathbb{R}^{2n}}[V_\Psi A(z,(M^T)^{-1}\zeta)v_s(M^{-1}z,\zeta)]\,\mathrm{d}\zeta\\ &= \int_{\mathbb{R}^{2n}}\sup_{z\in\mathbb{R}^{2n}}[V_\Psi A(z,\zeta)v_s(M^{-1}z,M^T\zeta)]\,\mathrm{d}\zeta.\end{aligned}$$

Diagonalizing M and using the rotational invariance of v_s it is easy to see that there exists a constant C_M such that

$$v_s(M^{-1}z, M^T\zeta) \leqslant C_M v_s(z,\zeta)$$

and hence the inequality (41). □

The modulation spaces $M_s^{\infty,1}(\mathbb{R}^{2n})$ contain many of the usual pseudo-differential symbol classes. For example is the vector space $C_b^{2n+1}(\mathbb{R}^{2n})$ of all functions which are differentiable up to order $2n+1$ with bounded derivatives. In fact, for every window Φ there exists a constant $C_\Phi > 0$ such that

$$\|A\|^{\Phi}_{M_s^{\infty,1}} \leqslant C_\Phi\|A\|_{C^{2n+1}} = C_\Phi\sum_{|\alpha|\leqslant 2n+1}\|\partial_z^\alpha A\|_\infty$$

and we have the inclusion

$$C_b^{2n+1}(\mathbb{R}^{2n}) \subset M_0^{\infty,1}(\mathbb{R}^{2n}). \tag{42}$$

3.2.1. The ∗-algebra and Wiener properties of $M_s^{\infty,1}(\mathbb{R}^{2n})$. Recall that the twisted product $A\#B$ of two symbols is defined by the formula

$$A\#B(z) = 4^n\int\int_{\mathbb{R}^{2n}}\mathrm{e}^{-4\pi\mathrm{i}\sigma(z-z',z-z'')}A(z')B(z'')\,\mathrm{d}z'\,\mathrm{d}z''. \tag{43}$$

For us the main interest of $M_s^{\infty,1}(\mathbb{R}^{2n})$ comes from the following property of the twisted product [24]:

Proposition 8. *Let $A, B \in M_s^{\infty,1}(\mathbb{R}^{2n})$. Then $A\#B \in M_s^{\infty,1}(\mathbb{R}^{2n})$. In particular, for every window Φ there exists a constant $C_\Phi > 0$ such that*

$$\|A\#B\|^{\Phi}_{M_s^{\infty,1}} \leqslant C_\Phi\|A\|^{\Phi}_{M_s^{\infty,1}}\|B\|^{\Phi}_{M_s^{\infty,1}}.$$

Since obviously $\overline{A} \in M_s^{\infty,1}(\mathbb{R}^{2n})$ if and only if $A \in M_s^{\infty,1}(\mathbb{R}^{2n})$ the property above can be restated as

The modulation space $M_s^{\infty,1}(\mathbb{R}^{2n})$ is a Banach ∗-algebra with respect to the twisted product # and the involution $A \longmapsto \overline{A}$.

The weighted Sjöstrand classes $M_s^{\infty,1}(\mathbb{R}^{2n})$ have the following property, see theorem 4.6 in [24]:

Proposition 9. *(i) Every Weyl operator* $\widehat{A}^{2\pi}$ *with* $A \in M_s^{\infty,1}(\mathbb{R}^{2n})$ *is bounded on* $L^2(\mathbb{R}^n)$*; (ii) if we have* $\widehat{C}^{2\pi} = \widehat{A}^{2\pi}\widehat{B}^{2\pi}$ *with* $A, B \in M_s^{\infty,1}(\mathbb{R}^{2n})$*, then* $C \in M_s^{\infty,1}(\mathbb{R}^{2n})$*; (iii) if* $\widehat{A}^{2\pi}$ *with* $A \in M_s^{\infty,1}(\mathbb{R}^{2n})$ *is invertible in* $L^2(\mathbb{R}^n)$ *with inverse* $\widehat{B}^{2\pi}$*, then* $B \in M_s^{\infty,1}(\mathbb{R}^{2n})$*.*

Property (i) thus extends the L^2-boundedness property of operators with symbols in $S^0_{0,0}(\mathbb{R}^{2n})$. Property (iii) is called the *Wiener property* of $M_s^{\infty,1}(\mathbb{R}^{2n})$; for the classical symbol classes results of this type go back to Beals [3]; in the recent paper by Gröchenig and Rzeszotnik [25], Beal's results are proven using Banach space methods.

We will apply this important result to deformation quantization in subsection 3.3.

3.3. Applications to deformation quantization

3.3.1. The $*$-algebra property for the Moyal product. Comparing formulae (5) and (43) we see that the twisted product is just the Moyal product with $\hbar = 1/2\pi$:

$$A\#B = A \star_{1/2\pi} B. \tag{44}$$

It turns out that more generally $A \star_\hbar B$ and $A\#B$ are related in a very simple way, and this has the following interesting consequence:

$$\text{If} \quad A, B \in M_s^{\infty,1}(\mathbb{R}^{2n}), \quad \text{then} \quad A \star_\hbar B \in M_s^{\infty,1}(\mathbb{R}^{2n}).$$

More precisely:

Proposition 10. *(i) The symbol class* $M_s^{\infty,1}(\mathbb{R}^{2n})$ *is a Banach $*$-algebra with respect to the Moyal product* $\star_\hbar$ *and the involution* $A \longmapsto \overline{A}$*: if A and B are in* $M_s^{\infty,1}(\mathbb{R}^{2n})$*, then* $A \star_\hbar B$ *is also in* $M_s^{\infty,1}(\mathbb{R}^{2n})$*. (ii) Let* $A \in M_s^{\infty,1}(\mathbb{R}^{2n})$ *and assume that* $A \star_\hbar B = I$*. Then* $B \in M_s^{\infty,1}(\mathbb{R}^{2n})$*.*

Proof. (i) Using the representation (9) of the Moyal product one sees immediately that

$$(A \star_\hbar B)_{\sqrt{\hbar}} = (A_{\sqrt{\hbar}})\#(B_{\sqrt{\hbar}}), \tag{45}$$

where $A_{\sqrt{\hbar}}(z) = A(z\sqrt{\hbar})$, etc. Since $M_s^{\infty,1}(\mathbb{R}^{2n})$ is a Banach $*$-algebra for the twisted convolution # it thus suffices to prove the equivalence

$$A_\lambda \in M_s^{\infty,1}(\mathbb{R}^{2n}) \quad \Longleftrightarrow \quad A \in M_s^{\infty,1}(\mathbb{R}^{2n}) \tag{46}$$

for every $\lambda > 0$. In fact, since $(A_\lambda)_{1/\lambda} = A$ it suffices to show that if $A \in M_s^{\infty,1}(\mathbb{R}^{2n})$ then $A_\lambda \in M_s^{\infty,1}(\mathbb{R}^{2n})$, which follows from proposition 7 for M to be the block diagonal matrix with entries λ and λ^{-1}. □

As a consequence we get the Wiener property for the Moyal product.

Corollary 11. *Suppose A is in* $M_s^{\infty,1}(\mathbb{R}^{2n})$ *and that there exists a B such that* $A \star_\hbar B = I$ *then* $A \in M_s^{\infty,1}(\mathbb{R}^{2n})$*.*

Proof. It immediately follows from proposition 10 above using the Wiener property of the twisted convolution (proposition 9(iii)). □

3.3.2. Regularity results for the star product. The following result combines the properties of the spaces $M_s^{\infty,1}(\mathbb{R}^{2n})$, viewed as symbol classes, with those of $M_s^q(\mathbb{R}^n)$.

Proposition 12. *Let $A \in M_s^{\infty,1}(\mathbb{R}^{2n})$. The operator $\widehat{A}^\hbar = A^w(x, -\mathrm{i}\hbar\partial_x)$ is bounded on $M_s^q(\mathbb{R}^n)$ for every q. In fact, there exists a constant $C > 0$ independent of q such that following uniform estimate holds*

$$\|\widehat{A}^\hbar\|_{M_s^q \longrightarrow M_s^q} \leqslant C\|A\|_{M_s^{\infty,1}}$$

for all $A \in M_s^{\infty,1}(\mathbb{R}^{2n})$.

Proof. The result is proven for $\hbar = 1/2\pi$ in [22], p 320 and p 323. Let us show that it holds for arbitrary $\hbar$. Noting that $A^w(x, -\mathrm{i}\hbar\partial_x) = B^w(x, -\mathrm{i}\partial_x)$ where $B(x, p) = A(x, 2\pi\hbar p)$ it suffices to show that if $A \in M_s^{\infty,1}(\mathbb{R}^{2n})$ then $B \in M_s^{\infty,1}(\mathbb{R}^{2n})$. But this follows from proposition 7 with the choice

$$M = \begin{pmatrix} I & 0 \\ 0 & 2\pi\hbar I \end{pmatrix}$$

for the change of variable. □

Note that if we take $q = 2, s = 0$ we have $M_0^2(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ hence operators with Weyl symbols in $M_s^{\infty,1}(\mathbb{R}^{2n})$ are bounded on $L^2(\mathbb{R}^n)$; in particular, using the inclusion (42), we recover the Calderón–Vaillancourt theorem [5].

We begin by making the following remark: there are elements of $L_s^q(\mathbb{R}^{2n})$ which do not belong to the range of any wave-packet transform W_ϕ (or, equivalently, to the range of any short-time Fourier transform $V_\phi\psi$). This is actually a somewhat hidden consequence of the uncertainty principle. Choose in fact a measurable function Ψ such that

$$\Psi(z) \leqslant C\,\mathrm{e}^{-\frac{1}{\hbar}Mz\cdot z},$$

where $C > 0$ and M is a real symmetric positive-definite matrix; clearly $\Psi \in L_s^q(\mathbb{R}^{2n})$, but the existence of ϕ and ψ such that $W_\phi\psi = \Psi$ is only possible if the matrix M satisfies the following very stringent condition (see [18, 19, 23]):

The moduli of the eigenvalues of JM are all $\leqslant 1$

which is equivalent to the geometric condition:

The section of the ellipsoid $Mz \cdot z \leqslant \hbar$ by any plane of conjugate coordinates x_j, p_j is $\geqslant \pi\hbar$.

The properties above are proven by using Hardy's uncertainty principle [27] which is a precise statement of the fact that a function and its Fourier transform cannot be simultaneously sharply localized. In the multi-dimensional case, this principle can be stated as follows [21]: if A and B are two real positive definite matrices and $\psi \in L^2(\mathbb{R}^n)$, $\psi \neq 0$ such that

$$|\psi(x)| \leqslant C_A \mathrm{e}^{-\frac{1}{2\hbar}Ax^2} \qquad \text{and} \qquad |F\psi(p)| \leqslant C_B\,\mathrm{e}^{-\frac{1}{2\hbar}Bp^2} \tag{47}$$

for some constants $C_A, C_B > 0$, then the eigenvalues λ_j, $j = 1, \ldots, n$, of AB are $\leqslant 1$. The statements above then follow, performing a symplectic diagonalization of M and using the marginal properties of the cross-Wigner transform.

We will call a function $\Psi \in L_s^q(\mathbb{R}^{2n})$ *admissible* if there exist $\psi \in M_s^q(\mathbb{R}^n)$ and a window ϕ such that $\Psi = W_\phi\psi$. Intuitively, the fact for a function to be admissible means that it is not 'too concentrated' around a phase-space point.

The modulation spaces $M_s^q(\mathbb{R}^n)$ can be used to prove the following regularity result in deformation quantization:

Proposition 13. *Assume that* $A \in M_s^{\infty,1}(\mathbb{R}^{2n})$ *and that* $B \in L_s^q(\mathbb{R}^{2n})$ *is admissible. Then* $A \star_\hbar B \in L_s^q(\mathbb{R}^{2n})$.

Proof. We have

$$A \star_\hbar B = \widetilde{A}^\hbar(B) = \widetilde{A}^\hbar(W_\phi \psi)$$

for some $\psi \in M_s^q(\mathbb{R}^n)$ and a window ϕ, and hence, using the first intertwining formula (28),

$$A \star_\hbar B = W_\phi(\widehat{A}^\hbar \psi).$$

Since $\psi \in M_s^q(\mathbb{R}^n)$ we have $W_\phi \psi \in L_s^q(\mathbb{R}^{2n})$ and proposition 12 implies that $\widehat{A}^\hbar \psi \in M_s^q(\mathbb{R}^n)$ hence $W_\phi(\widehat{A}^\hbar \psi) \in L_s^q(\mathbb{R}^{2n})$ which we set out to prove. □

3.3.3. The star exponential. Recently, we pointed out that there is an approach to the Moyal quantization in terms of the Bopp-calculus [20]. We continue this line of research in this subsection.

Let H be a Hamiltonian function. In deformation quantization one defines the star-exponential $\exp(Ht)$ by the formal series

$$\exp(Ht) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{t}{\mathrm{i}\hbar}\right)^k (H\star_\hbar)^k,$$

where $(H\star_\hbar)^0 = \mathrm{Id}$ and $(H\star_\hbar)^k = H \star_\hbar (H\star_\hbar)^{k-1}$ for $k \geqslant 1$. In terms of the Bopp pseudo-differential operator $\widetilde{H}$ we thus have

$$\exp(Ht) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{t}{\mathrm{i}\hbar}\right)^k \widetilde{H}^k, \tag{48}$$

this formula allows us to *redefine* the star-exponential $\exp(Ht)$ by

$$\exp(Ht) = \exp\left(-\frac{\mathrm{i}t}{\hbar}\widetilde{H}\right). \tag{49}$$

With this redefinition $\exp(Ht)$ is the evolution operator for the phase-space Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial \Psi}{\partial t} = \widetilde{H}\Psi, \qquad \Psi(\cdot, 0) = \Psi_0. \tag{50}$$

That is, the solution Ψ of the Cauchy problem (50) is given by

$$\Psi(z, t) = \exp(Ht)\Psi_0(z). \tag{51}$$

Let now

$$U_t = \exp\left(-\frac{\mathrm{i}t}{\hbar}\widehat{H}^\hbar\right) \tag{52}$$

be the evolution operator for the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(x, t) = \widehat{H}^\hbar\psi(x, t), \qquad \psi(x, 0) = \psi_0(x) \tag{53}$$

with Hamiltonian operator $\widehat{H}$. (We will always assume that the solutions to (53) exist for all t and are unique for an initial datum $\psi_0 \in \mathcal{S}(\mathbb{R}^n)$.)

The following intertwining and conjugation relations are obvious:

$$\exp(Ht)W_\phi = W_\phi U_t \tag{54}$$

$$W_\phi^* \exp(Ht) = \exp U_t W_\phi^* \tag{55}$$

$$W_\phi^* \exp(Ht) W_\phi = \exp U_t. \tag{56}$$

We also note that it immediately follows from formula (18) in proposition 2 that we have the symplectic covariance formula

$$\exp[(H \circ S^{-1})t] = U_S \exp(Ht) U_S^{-1}$$

where $U_S \in \mathrm{Mp}(4n, \mathbb{R})$ is defined by

$$U_S \Psi(z) = \Psi(Sz)$$

for $S \in \mathrm{Sp}(2n, \mathbb{R})$.

The following result shows that the star exponential preserves the admissible functions in the weighted L^q spaces:

Proposition 14. *Assume that the Hamiltonian is of the type*

$$H(z) = \tfrac{1}{2} Mz \cdot z + m \cdot z \tag{57}$$

where M is symmetric and $m \in \mathbb{R}^{2n}$. Let $\Psi \in L_s^q(\mathbb{R}^{2n})$ be admissible. Then

$$\exp(Ht)\Psi \in L_s^q(\mathbb{R}^{2n}) \qquad \text{for all} \quad t \in \mathbb{R} \tag{58}$$

for all $q \geqslant 1$ and $s \geqslant 0$.

Proof. Assume first that $m = 0$; then the Hamiltonian flow determined by H consists of symplectic matrices and is thus a one-parameter subgroup (S_t) of $\mathrm{Sp}(2n, \mathbb{R})$. To (S_t) there corresponds a unique one-parameter subgroup $\left(\widehat{S_t^\hbar}\right)$ of the metaplectic group $\mathrm{Mp}(2n, \mathbb{R})$, and we have $U_t = \widehat{S_t^\hbar}$, that is, the function $\psi(x,t) = \widehat{S_t^\hbar}\psi_0(x)$ is the solution to Schrödinger's equation (53) (see, for instance [16], chapter 7, section 7.2.2). In view of proposition 4(ii) we have $\widehat{S_t^\hbar} : M_s^q(\mathbb{R}^n) \longrightarrow M_s^q(\mathbb{R}^n)$. If Ψ is admissible there exists $\psi \in M_s^q(\mathbb{R}^n)$ and a window ϕ such that $\Psi = W_\phi \psi$ hence, taking formula (54) into account,

$$\exp(Ht)\Psi = W_\phi U_t \psi;$$

since $U_t\psi \in M_s^q(\mathbb{R}^n)$ we have $W_\phi U_t \psi \in L_s^q(\mathbb{R}^{2n})$ hence (58) when $m = 0$. The case $m \neq 0$ follows since the one-parameter subgroup (S_t) of $\mathrm{Sp}(2n, \mathbb{R})$ is replaced by a one-parameter subgroup of the inhomogeneous (=affine) symplectic group $\mathrm{ISp}(2n, \mathbb{R})$, from which follows that $U_t = \widehat{S_t^\hbar}$ is replaced by $U_t = \widehat{S_t^\hbar}\widehat{T^\hbar}(z_0)$ for some $z_0 \in \mathbb{R}^{2n}$ only depending on m (see [29]); one concludes exactly as above using the invariance of $M_s^q(\mathbb{R}^n)$ under the action of Weyl–Heisenberg operators (proposition 4(i)). □

4. Concluding remarks

We want to briefly comment on another approach to the Moyal $*$-product, namely on the work of Dubois-Violette, Kriegl, Maeda and Michor about smooth Heisenberg planes. In [9] the authors introduced *smooth $*$-algebras*, i.e. involutive algebras having many derivations. Their main examples are the smooth noncommutative torus and the smooth Heisenberg algebra. In the discussion of these examples the twisted convolution and its relation to Moyal $*$-product is of central importance. Their results rely on the spaces $\mathcal{O}_c(\mathbb{R}^n)$, $\mathcal{O}_m(\mathbb{R}^n)$, $\mathcal{O}_c'(\mathbb{R}^n)$ and $\mathcal{O}_m'(\mathbb{R}^n)$. We briefly recall the definition of these spaces.

The space $\mathcal{O}_c(\mathbb{R}^n)$ is the space of all smooth functions f on $\mathbb{R}^n$ for which there exists $k \in \mathbb{Z}$ such that $x \mapsto v_k \cdot f(x) = (1 + |x|^2)^{k/2} f(x)$ is bounded for each multiindex $\alpha \in \mathbb{N}_0^n$. The dual space $\mathcal{O}_c'(\mathbb{R}^n)$ is called the space of rapidly decreasing distributions.

The space $\mathcal{O}_m(\mathbb{R}^n)$ consists of all smooth functions on $\mathbb{R}^n$ such that for each multiindex $\alpha \in \mathbb{N}_0^n$ there exists $k \in \mathbb{Z}$ such that $x \mapsto v_k \cdot \partial^\alpha f(x) = (1 + |x|^2)^k \partial^\alpha f(x)$ is bounded.

$\mathcal{O}_m(\mathbb{R}^n)$ is the space of tempered functions on $\mathbb{R}^n$ and its dual space $\mathcal{O}'_m(\mathbb{R}^n)$ is known as the space of speedily decreasing distributions. Important facts about these spaces are that $\mathcal{O}_m(\mathbb{R}^n) \cong \mathcal{O}'_c(\mathbb{R}^n)$ and

$$\mathcal{S} \subset \mathcal{O}_c \subset \mathcal{O}_m \subset \mathcal{S}', \qquad \mathcal{S} \subset \mathcal{O}'_m \subset \mathcal{O}'_c \subset \mathcal{S}'. \tag{59}$$

A great part of the work in [9] is to understand the properties of twisted convolution on these spaces. Since Dubois-Violette Kriegl, Michor and Maeda use the definition of these space as noted above, i.e. in terms of differentiation, it is not clear at all how their results relate to those about $M_s^{\infty,1}(\mathbb{R}^{2n})$. We are convinced that a description in terms of STFT or Wigner transform of the spaces $\mathcal{O}_c(\mathbb{R}^n)$, $\mathcal{O}_m(\mathbb{R}^n)$, $\mathcal{O}'_c(\mathbb{R}^n)$ and $\mathcal{O}'_m(\mathbb{R}^n)$ would be of great help in clarifying the connection between their approach and the one taken here. Currently, we just know that $M_s^{\infty,1}(\mathbb{R}^{2n})$ is a subspace of $\mathcal{S}'(\mathbb{R}^{2n})$ like $\mathcal{O}'_c(\mathbb{R}^{2n})$ but we have no idea about the relation between the spaces $M_s^{\infty,1}(\mathbb{R}^{2n})$ and $\mathcal{O}'_c$. Finally we want to emphasize that $M_s^{\infty,1}(\mathbb{R}^{2n})$ is a Banach algebra with respect to twisted convolution and therefore these spaces are much easier to handle than the locally convex spaces in [9].

Our results are not the most general possible. The spaces considered in this paper are members of the class of modulation spaces. Our choice was dictated by the fact that while many of the results we have stated still remain valid for these more general modulation spaces if certain natural assumptions (for instance sub-additivity) are made on the weight m the notation can sometimes appear as too complicated. Another topic we only briefly mentioned is the Feichtinger algebra $M_0^1(\mathbb{R}^n) = S_0(\mathbb{R}^n)$. In addition to the properties we listed, it has the following nice feature: let $S'_0(\mathbb{R}^n)$ be the dual of $S_0(\mathbb{R}^n)$, then $(S_0(\mathbb{R}^n), L^2(\mathbb{R}^n), S'_0(\mathbb{R}^n))$ is a Gelfand triple of Banach spaces; this property makes $S_0(\mathbb{R}^n)$ particularly adequate for the study of the continuous spectrum of operators.

Another direction certainly worth to be explored is the theory of Wiener amalgam spaces [10, 22], which are closely related to modulation spaces; Cordero and Nicola [7] have obtained very interesting results for the Schrödinger equation using Wiener amalgam spaces. What role do they play in deformation quantization?

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